九十八學年度台灣省第八區(高屏區)

高級中學數理及資訊學科能力競賽

（數學科筆試二參考解答）

一、 令 $x=1=y$

$[f(0)]^2-f(0)+2=0 \quad \therefore\ f(0)=2 \quad \text{or} \quad f(0)=-1$

令 $y=1$

(i) $f(0)=2$

$\therefore\ f(x-1)=1+x \qquad\qquad \therefore\ f(2009)=2011$

(ii) $f(0)=-1$

$\therefore\ f(x-1)=1-2x \qquad\qquad \therefore\ f(2009)=1-2\times 2010=-4019$

二、 $(x^2+1)^{200}=((x^2+1)^2)^{100}$

$$
\begin{aligned}
&=(x^4+1+2x^2)^{100}\\
&=\sum_{i=1}^{100} c_i^{100}(x^4+1)^i(2x^2)^{100-i}+(2x^2)^{100}\\
&=A+2^{100}(x^4+1-1)^{50}\\
&=A+B+2^{100}(-1)^{50}
\end{aligned}
$$

其中 A,B 可被 x^4+1 整除，所以餘式為 2^{100}。

三、

$$
\begin{aligned}
&\frac{p}{q}+\frac{p}{r}+\frac{q}{p}+\frac{q}{r}+\frac{r}{p}+\frac{r}{q}\\
&=\frac{p^2r+p^2q+q^2r+q^2p+r^2q+r^2p}{pqr}\\
&=\frac{p(pr+pq+qr)+q(qr+pq+pr)+r(rq+pr+pq)-3pqr}{pqr}\\
&=\frac{(pr+pq+qr)(p+q+r)-3pqr}{pqr}\\
&=\frac{3\times 5-3\times 2}{2}\\
&=\frac{9}{2}
\end{aligned}
$$

四、

$$\overrightarrow{AM}=\frac{1}{2}\overrightarrow{AP}=\frac{1}{2}(\frac{1}{3}\overrightarrow{AB}+\frac{2}{3}\overrightarrow{AC})$$

$$=\frac{1}{6}\overrightarrow{AB}+\frac{1}{3}\overrightarrow{AC}$$

令 AB / AD = s, AC /AE = t

$$\overrightarrow{AM}=\frac{s}{6}\overrightarrow{AD}+\frac{t}{3}\overrightarrow{AE}$$

$s/6 + t/3 = 1$

$s + 2t = 6$

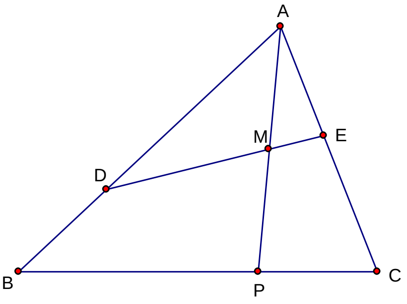
$\triangle ABM/\triangle ADM = s,\ 1/a = s/\triangle ABM$

$\triangle ACM/\triangle AEM = t,\ 1/b = t/\triangle ACM$

$\triangle ABM = \triangle ABP/2 = \triangle ABC/3$

$\triangle ACM = \triangle ACP/2 = \triangle ABC/6$

$1/a + 1/b = 3s/c + 6t/c = (s + 2t)3/c = 18/c$



五、

$$\left[\sin\alpha\cdot(1-\tan\beta)+\cos\alpha\cdot(1+\tan\beta)\right]^2\le\left(\sin^2\alpha+\cos^2\beta\right)\cdot\left[(1-\tan\beta)^2+(1+\tan\beta)^2\right]$$

$$=2\left(1+\tan^2\beta\right)=2\sec^2\beta$$

$\Rightarrow\ \sin\alpha\cdot(1-\tan\beta)+\cos\alpha\cdot(1+\tan\beta)\le\sqrt{2}\sec\beta$，且

當"="成立時 $\Leftrightarrow\ \dfrac{\sin\alpha}{1-\tan\beta}=\dfrac{\cos\alpha}{1+\tan\beta}$

$\Leftrightarrow\ \tan\alpha=\dfrac{1-\tan\beta}{1+\tan\beta}=\tan\left(\dfrac{\pi}{4}-\beta\right)$

因 $\alpha,\beta\in\left(0,\frac{\pi}{4}\right)$，故 $\alpha+\beta=\frac{\pi}{4}$。