八十八學年度高級中學數學能力競賽試題(二)參考解答(屏東高

1.參考解答

設 C 點的座標為$(\alpha，\alpha^2)$，$0\le\alpha\le1$

D 點的座標為$(x_0，y_0)$

解$\begin{cases} y=\alpha+(\alpha-1)x \\ x^2+y^2=2 \end{cases}$

得 $x_0=\dfrac{2-\alpha^2}{\alpha^2-2\alpha+2}$　　$y_0=\dfrac{-\alpha^2+4\alpha-2}{\alpha^2-2\alpha+2}$

$$\overline{CD}^2=(x_0-\alpha)^2+(y_0-\alpha^2)^2$$

$$=x_0{}^2+y_0{}^2-2\alpha^2x_0-2\alpha^2y_0+\alpha^2+\alpha^4$$

$$=2+\alpha^2+\alpha^4-2\alpha x_0-2\alpha^2y_0$$

$$\overline{BC}^2=(1-\alpha)^2+(1-\alpha^2)^2=2+\alpha^2+\alpha^4-2\alpha-2\alpha^2$$

$$\overline{CD}^2-\overline{BC}^2=2\alpha(1-x_0)+2\alpha^2(1-y_0)$$

$$=2\alpha\cdot\frac{(2\alpha^2-2\alpha)}{\alpha^2-2\alpha+2}+2\alpha^2\cdot\frac{2\alpha^2-6\alpha+4}{\alpha^2-2\alpha+2}$$

$$=\frac{4\alpha^2(\alpha-1+\alpha^2-3\alpha+2)}{\alpha^2-2\alpha+2}=\frac{4\alpha^2(\alpha^2-2\alpha+1)}{\alpha^2-2\alpha+2}\ge0$$

2.參考解答

(1) $\overline{A_1A_2}=2$(半徑× $\sin15^\circ$)$=4\times\dfrac{\sqrt6-\sqrt2}{4}=\sqrt6-\sqrt2$

(2) $\overline{A_1A_2}+\overline{A_1A_3}+\ldots\ldots+\overline{A_1A_{12}}$

$=4(\sin15^\circ+\sin30^\circ+\cdots\cdots+\sin165^\circ)$

$=4(\sqrt6+\sqrt3+\sqrt2+2)$

令 $w=\cos15^\circ+i\sin15^\circ=\cos\dfrac{\pi}{12}+i\sin\dfrac{\pi}{12}$

$$w+w^2+w^3+\ldots\ldots+w^{11}=\frac{w-w^{12}}{1-w}=\frac{\cos15^\circ+i\sin15^\circ-(-1)}{1-(\cos15^\circ+i\sin15^\circ)}$$

$$=\frac{\cos15^\circ+i\sin15^\circ+1}{(1-\cos15^\circ)-i\sin15^\circ}\times\frac{(1-\cos15^\circ)+i\sin15^\circ}{(1-\cos15^\circ)+i\sin15^\circ}$$

$$=\frac{1-i\sin15^\circ}{1-i\cos15^\circ}=i(\sqrt6+\sqrt3+\sqrt2+2)$$

3.參考解答

198319831983 能被 7 整除

2000÷ 3 的餘數為 2

即為 19831983÷ 7 的餘數為 3

3+4=7，同此為星期日

4.參考解答

設 f(x)之實根的 $\alpha_1 \cdots\cdots \alpha_n$，則 $f(x)=a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_1 x + a_0$

$= a_n(x-\alpha_1)(x-\alpha_2) + \ldots\ldots + (x-\alpha_3)$

故 $\alpha_1 + \ldots\ldots + \alpha_n = \dfrac{-a_{n-1}}{a_n}$ 且 $\displaystyle\sum_{1\le i\le j\le n} \alpha_1\alpha_j = \frac{a_{n-2}}{a_n}$

$\dfrac{a_{n-1}^2}{a_n^2} = (\alpha_1 + \ldots + \alpha_n)^2 = \alpha_1^2 + \ldots\ldots + \alpha_1^2 + 2\displaystyle\sum_{1\le i<j\le n} \alpha_i\alpha_j \ge 2\cdot\frac{a_{n-2}}{a_n}$

$\therefore a_{n-1}^2 \ge 2a_{n-2}a_n$　　圓形(或利用倒根變換) $a_1^2 \ge 2a_0a_n$

5.參考解答

令 $s=\cos\dfrac{2\pi}{7}+\sin\dfrac{2\pi}{7}$　故 $s^7=1$　因 $s\neq 1$，故 $s^6+s^5+\ldots\ldots+s=1$

$1=(s^6+s)+(s^5+s^2)+(s^4+s^3)=2\cos\dfrac{2\pi}{7}+2\cos\dfrac{4\pi}{7}+2\cos\dfrac{6\pi}{7}$

$=2\cos\dfrac{2\pi}{7}+2(2\cos^2\dfrac{2\pi}{7}-1)+2(3\cos^3\dfrac{2\pi}{7}-4\cos\dfrac{2\pi}{7})$

整理之，故 $2\cos\dfrac{2\pi}{7}$ 為 x^3+x^2-2x-1 之一根

6.參考解答

設 p 點的座標為$(\cos\theta, \sin\theta)$，$0<\theta<\dfrac{\pi}{2}$

$\overline{PA}+\overline{PB}$

$=\sqrt{(1-\cos\theta)^2+\sin^2\theta}+\sqrt{\cos^2+(1-\sin\theta)^2}=\sqrt{2-2\cos\theta}+\sqrt{2-2\sin\theta}$

$=2\cdot\sin\dfrac{\theta}{2}+\sqrt{2}(\cos\dfrac{\theta}{2}-\sin\dfrac{\theta}{2})$

$=(2-\sqrt{2})\sin\dfrac{\theta}{2}+\sqrt{2}\cos\dfrac{\theta}{2}$

$=2\sqrt{2-\sqrt{2}}\cos(\dfrac{\theta}{2}-\beta)$其中$\cos\beta=\dfrac{\sqrt{2}}{2\sqrt{2-\sqrt{2}}}=\dfrac{\sqrt{2+\sqrt{2}}}{2}$　　　$\beta=\dfrac{\pi}{8}$

當 $\theta=\dfrac{\pi}{4}$時，$\overline{PA}+\overline{PB}$有最大值，此時四邊形 OAPB 的周長有最大值